Coordinate changes for Integrals

In Calc I we used coord. changes to solve

$$\int_{x=0}^{5} x e^{x^2} dx \quad \underbrace{u = x^2 \quad du = 2x\,dx}_{}$$ Coordinate change parameratizes $\mathbb{R}$

Necessary computation for new coordinates

Polar Coordinate change

$$\iint_R e^{x^2+y^2} dA = \iint_{R_{polar}} e^{r^2} r\, dA_{polar} \quad \begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

Want: A more general way to compute these Coordinate Changes for integrals (i.e. compute the differentials, need to make differential computations easier)

How do we do that: Jacobian

Def: The signed Jacobian of a coordinate change

$$\begin{cases} x_1 = x_1(u_1, u_2, \cdots, u_n) \\ x_2 = x_2(u_1, u_2, \cdots, u_n) \\ \vdots \\ x_3 = x_3(u_1, u_2, \cdots, u_n) \end{cases} \Rightarrow \frac{\partial(x_1, x_2, \cdots x_n)}{\partial(u_1, u_2, \cdots, u_n)}$$

Writting each variable in terms of the parameratizing variable(s)

$$\frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(u_1, u_2, \cdots, u_n)} = \det\begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{bmatrix}$$

The Signed Jacobian of the Polar coord. change $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

$$= \cos(\theta)(r\cos(\theta)) - (-r\sin(\theta))\sin(\theta) = r(\cos^2(\theta)+\sin^2(\theta)) = r$$

If we reverse the order of r & θ we get a different thing

$$\frac{\partial(x,y)}{\partial(\theta,r)} = \det\begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial r} \end{bmatrix} = \begin{bmatrix} -r\sin(\theta) & \cos(\theta) \\ r\cos(\theta) & \sin(\theta) \end{bmatrix}$$

$$= -r\sin(\theta)(\sin(\theta)) - (r\cos(\theta))\cos(\theta) = -r(\cos^2(\theta)+\sin^2(\theta)) = -r$$

Def: Unsigned Jacobian of a transformation is just the absolute value of the Jacobian

$$\left|\frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(u_1, u_2, \cdots, u_n)}\right|$$

Prop: Let $f(x_1, x_2, \cdots, x_n)$ be a function continuous on $\mathbb{R}$ and

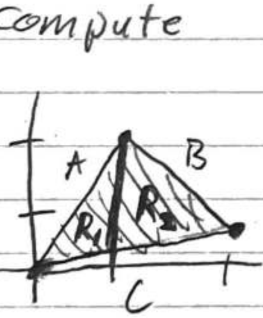
$$\begin{cases} x_1 = x_1(u_1, u_2, \cdots, u_n) \\ \vdots \\ x_n = x_n(u_1, u_2, \cdots, u_n) \end{cases}$$

Be a coordinate change by differentiable functions then

$$\int_{R_{old}} f(x_1, \cdots, x_n)\, dV_{old} = \int_{R_{new}} f(x_1(u_1, \cdots, u_n), \cdots, x_n(u_1, \cdots, u_n)) \left| \frac{\partial(x_1, \cdots, x_n)}{\partial(x_n(u_1, \cdots, u_n))} \right| dV_{new}$$

Ex: $\iint_R (x-2y)\,dA$ for R the triangle with verticies $(0,0), (1,2), (2,1)$

Sol 1: In cartesian coordinates, split region and compute



Side A: $m_A = \frac{2-0}{1-0} = 2 \quad y-0 = 2(x-0) \quad y = 2x$

Side B: $m_B = \frac{1-2}{2-1} = -1 \quad y-2 = -1(x-1) \quad y = 3-x$

Side C: $m_C \ \frac{1-0}{2-0} = \frac{1}{2} \quad y-0 = \frac{1}{2}(x-0) \quad y = \frac{1}{2}x$

$$\iint_R (x-2y)\,dA = \iint_{R_1} (x-2y)\,dA + \iint_{R_2} (x-2y)\,dA$$

$$= \int_{x=0}^{1} \int_{y=\frac{x}{2}}^{2x} (x-2y)\,dy\,dx + \int_{x=1}^{2} \int_{y=\frac{x}{2}}^{3-x} (x-2y)\,dy\,dx$$

$$\int_{x=0}^{1}\int_{y=\frac{x}{2}}^{2x}(x-2y)\,dy\,dx = \int_{x=0}^{1}\left(xy-y^2\Big|_{y=\frac{x}{2}}^{2x}\right)dx = \int_{x=0}^{1}\left(2x^2-4x^2-\frac{x^2}{2}+\frac{x^2}{4}\right)dx$$

$$=\int_{x=0}^{1}\left(-2x^2-\frac{x^2}{4}\right)dx = \int_{x=0}^{1}\frac{-9}{4}x^2\,dx = \frac{-9}{12}(1-0) = \frac{-3}{4}$$

$$\int_{x=1}^{2}\int_{y=\frac{x}{2}}^{3-x}(x-2y)\,dy\,dx = \int_{x=1}^{2}\left(xy-y^2\Big|_{y=\frac{x}{2}}^{3-x}\right)dx$$

$$=\int_{x=1}^{2}\left(x(3-x)-(3-x)^2-\frac{x^2}{2}+\frac{x^2}{4}\right)dx$$

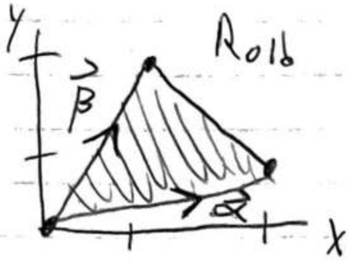
$$=\int_{x=1}^{2}\left(3x-x^2-(9-6x+x^2)-\frac{x^2}{4}\right)dx = \int_{x=1}^{2}\left(-\frac{9}{4}x^2+9x-9\right)dx$$

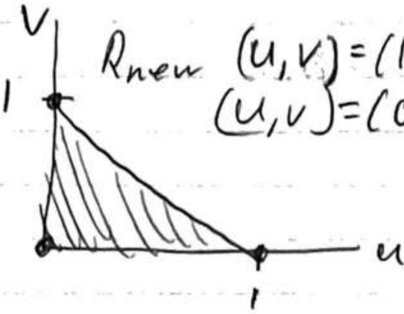
$$=\left(-\frac{9}{12}x^3+\frac{9}{2}x^2-9x\Big|_{x=1}^{2}\right) = -\frac{3}{4}\cancel{(2)^3}^{\,7}+\frac{9}{2}\cancel{(2)^2}^{\,3}-9(2)+\cancel{\frac{3}{4}(1)^3}-\cancel{\frac{9}{2}(1)^2}+\cancel{9}$$

$$=\frac{-21}{4}+\frac{27}{2}-9=\frac{-3}{4}$$

$$\iint_R (x-2y)\,dA = \iint_{R_1}(x-2y)\,dA + \iint_{R_2}(x-2y)\,dA = \frac{-3}{4}+\frac{-3}{4} = \frac{-3}{2}$$

Sol 2: Use Transformation





$(u,v)=(1,0) \rightsquigarrow (x,y)=(2,1)$
$(u,v)=(0,1) \rightsquigarrow (x,y)=(1,2)$

$v = 1-u$

$(x,y) = u\vec{\alpha} + v\vec{\beta}$

Using the trasformation $\begin{cases} x = 2u+v \\ y = u+2v \end{cases}$

Check: $(x,y)\big|_{\substack{u=1\\v=0}} = (2,1)$, $(x,y)\big|_{\substack{u=0\\v=1}} = (1,2)$, $(x,y)\big|_{\substack{u=0\\v=0}} = (0,0)$

Moreover $\{(u,v) \mid 0 \le u \le 1, 0 \le v \le 1-u\} = R_{new}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 4-1 = 3$$

$$\iint_{R_{old}} (x-2y)\,dA = \iint_{R_{new}} (2u+v-2(u+2v))\,|3|\,dA_{new}$$

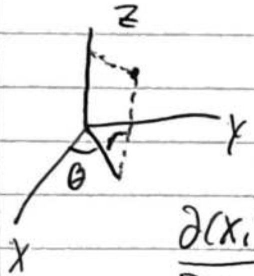
$$\int_{u=0}^{1}\int_{v=0}^{1-u} -9v\,dv\,du = \int_{u=0}^{1}\left(-\frac{9}{2}v^2\Big|_{v=0}^{1-u}\right)du = -\frac{9}{2}\int_{u=0}^{1}(1-u)^2\,du$$

$$= \frac{9}{6}\left((1-u)^3\Big|_{u=0}^{1}\right) = \frac{3}{2}(0-1) = -\frac{3}{2}$$

Generalizing Polar Coordinates to $\mathbb{R}^3$

Cylindrical Coordinates (Naive)

Idea: Paramaterize a plane and leave the other axis alone



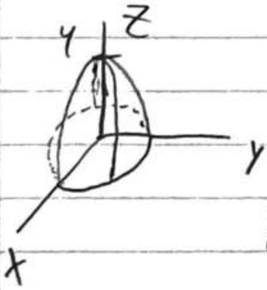
In Particular $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \det\begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \cos(\theta)(r\cos(\theta) - 0) - (-r\sin(\theta))(\sin(\theta) - 0)) + 0(0-0)$$
$$= r\cos^2(\theta) + r\sin^2(\theta) = r$$

$dA_{cart} = r\,dA_{cylinder}$ for all "Standard" cylindrical trasformations

Ex: Compute $\iiint_E (x+y+z)\,dV$ for E in the first octant and below the paraboloid $4 - x^2 - y^2 = z$



Using $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$ $4 - (x^2+y^2) = z = 4 - r^2$

when $z = 0$, $4 - r^2 = 0$ $r = \pm 2$

$$E = \left\{(r,\theta,z) \,\middle|\, 0 \le \theta \le \tfrac{\pi}{2},\ 0 \le r \le 2,\ 0 \le z \le 4 - r^2\right\}$$

$$\iiint_E (x+y+z)\,dV = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2} \int_{z=0}^{4-r^2} (r\cos(\theta) + r\sin(\theta) + z)\,r\,dz\,dr\,d\theta$$

$$= \int_{r=0}^{2}\int_{z=0}^{4-r^2}\int_{\theta=0}^{\frac{\pi}{2}} (r^2\cos(\theta) + r^2\sin(\theta) + zr)\, d\theta\, dz\, dr$$

$$= \int_{r=0}^{2}\int_{z=0}^{4-r^2} \left(r^2(\sin(\theta) - \cos(\theta)) + zr\theta \Big|_{\theta=0}^{\frac{\pi}{2}} \right) dz\, dr$$

$$= \int_{r=0}^{2}\int_{z=0}^{4-r^2} \left(r^2(1-0) + \frac{zr\pi}{2} - r^2(0-1) - 0 \right) dz\, dr$$

$$= \int_{r=0}^{2}\int_{z=0}^{4-r^2} \left(2r^2 + \frac{zr\pi}{2} \right) dz\, dr = \int_{r=0}^{2} \left(2r^2 z + \frac{z^2 r\pi}{4} \Big|_{z=0}^{4-r^2} \right) dr$$

$$= \int_{r=0}^{2} \left(2r^2(4-r^2) + \frac{r\pi}{4}(4-r^2)^2 \right) dr = \int_{r=0}^{2} \left(8r^2 - 2r^4 + \frac{\pi}{4}(16r - 8r^3 + r^5) \right) dr$$

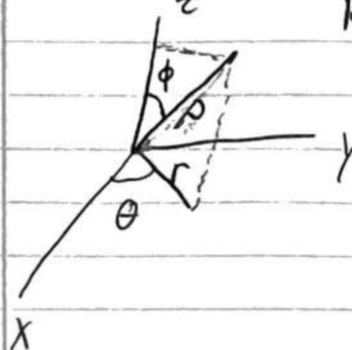
$$= \left(\frac{8}{3}r^3 - \frac{2}{5}r^5 + \frac{\pi}{4}\left(8r^2 - 2r^4 + \frac{r^6}{6}\right) \Big|_{r=0}^{2} \right)$$

$$= \frac{8}{3}(2)^3 - \frac{2}{5}(2)^5 + \frac{\pi}{4}\left(8(2)^2 - 2(2)^4 + \frac{2^6}{6}\right) = \frac{64}{3} - \frac{64}{5} + \frac{\pi}{4}\left(\cancel{32} - \cancel{32} + \frac{64}{6}\right)$$

$$= \frac{64}{3} - \frac{64}{5} + \frac{8\pi}{3}$$

Sherical Coordinates: Every point in $\mathbb{R}^3$ lives on a shere

Reparamaterizing



ρ = Distance from (x,y,z) to origin

θ = Angle from positive x-axis to point $(x,y,0)$

ϕ = Angle from positive z-axis to point (x,y,z)

$$\begin{cases} x = r\cos(\theta) = \rho\sin(\phi)\cos(\theta) \\ y = r\sin(\theta) = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases} \qquad \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2\sin(\phi)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \det\begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{bmatrix} = \det\begin{bmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{bmatrix}$$

$$= \sin(\phi)\cos(\theta)\left(-\rho^2\sin^2(\phi)\cos(\theta) - 0\right) - \sin(\phi)\sin(\theta)\left(\rho^2\sin^2(\phi)\sin(\theta) - 0\right)$$
$$+\cos(\phi)\left(-\rho^2\sin(\phi)\cos(\phi)\sin^2(\theta) - \rho^2\sin(\phi)\cos(\phi)\cos^2(\theta)\right)$$

$$= -\rho^2\sin^3(\phi)\cos^2(\theta) - \rho^2\sin^3(\phi)\sin^2(\theta) - \rho^2\sin(\phi)\cos^2(\phi)\sin^2(\theta)$$
$$-\rho^2\sin(\phi)\cos^2(\phi)\cos^2(\theta)$$

$$= -\rho^2\sin^3(\phi)\overset{1}{\cancel{(\cos^2(\theta)+\sin^2(\theta))}} - \rho^2\sin(\phi)\cos^2(\phi)\overset{1}{\cancel{(\sin^2(\theta)+\cos^2(\theta))}}$$

$$= -\rho^2\sin(\phi)\overset{1}{\cancel{(\sin^2(\phi)+\cos^2(\phi))}} = -\rho^2\sin(\phi)$$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin(\phi)$$